Advanced Microeconomics

Lecture 4: Strategic Interaction, Nash Equilibrium, and Sequential Games

Yihu Hou

Renmin University of China

Spring 2026

Outline

1. Why oligopoly pushes us into game theory
2. Normal-form games: best responses, dominance, and Nash equilibrium
3. Sequential games: extensive form, backward induction, and subgame perfection
4. Applications: bargaining and repeated interaction

From Chapter 2 to Chapter 3

Perfect competition

Each firm is too small to affect the market price.

$$\max_q \; pq - C(q)$$

The firm takes p as given.

Cournot / Bertrand

Each firm chooses while anticipating rivals' choices.

$$\pi_i(a_i, a_{-i})$$

Payoff depends on *all* actions.

Stackelberg

Timing also matters:

leader moves first, follower reacts.

The relevant question is no longer only "what is optimal?" but also "optimal given others' reactions?"

The new object

Once each player's best action depends on what others do, the right language is a **game**, not a single supply curve.

What every game must specify

Core ingredients

A game tells us:

1. who the **players** are,
2. what each player can choose,
3. how payoffs depend on all choices.

In compact notation:

$$G = \big(N, (S_i)_{i \in N}, (u_i)_{i \in N}\big).$$

Timing and information

The same payoff functions can lead to different outcomes if:

- players move **at the same time**,
- one player moves **after observing** another,
- or some actions / types are hidden.

Important reading rule

"Move at the same time" means **choose without observing the rival's choice**, not necessarily that the physical actions happen in the same second. Sealed bids are the classic example.

Action, strategy, and solution concept

In a simultaneous-move game

Usually each player chooses once.

- action = strategy
- solution concept: **Nash equilibrium**

In a sequential game

A strategy is a **complete contingent plan**:

- what would you do after every history that might arise?

Example

Cournot: choose quantity once. Bertrand: choose price once.

Example

Entry game: "If entry occurs, fight" is part of a strategy even if entry never happens on the realized path.

Message

The correct solution concept must match the timing structure:

$$\text{simultaneous play} \Rightarrow \text{Nash}, \qquad \text{sequential play} \Rightarrow \text{backward induction / SPE.}$$

Normal-form representation

Strategic form

For each player i:

- strategy set S_i,
- payoff function $u_i(s_1, \ldots, s_n)$.

A 2-player matrix

	L	R
U	$(3, 2)$	$(0, 1)$
D	$(1, 0)$	$(2, 3)$

How to read it

- Player 1 chooses rows.
- Player 2 chooses columns.
- Each cell gives the payoff pair (u_1, u_2).

Main question

Which cell is stable once each player correctly anticipates the other's choice?

Best responses and pure Nash equilibrium

	L	R
U	$(3, 2)$	$(0, 1)$
D	$(1, 0)$	$(2, 3)$

teal: Player 1's best response   orange: Player 2's best response

Best response

$$\mathrm{BR}_i(s_{-i}) = \arg\max_{s_i \in S_i} u_i(s_i, s_{-i}).$$

- If Player 2 chooses L, Player 1 prefers U.
- If Player 2 chooses R, Player 1 prefers D.
- If Player 1 chooses U, Player 2 prefers L.
- If Player 1 chooses D, Player 2 prefers R.

Pure Nash equilibria

Cells where *both* players are best-responding:

$$(U, L), \qquad (D, R).$$

Dominance is stronger than best response

Strict dominance

Strategy s_i **strictly dominates** s_i' if

$$u_i(s_i, s_{-i}) > u_i(s_i', s_{-i}) \quad \forall s_{-i}.$$

Interpretation

Best response is a **local** comparison: "best against this particular rival action."
Dominance is a **global** comparison: "better no matter what the rival does."

Why dominance matters

- A strictly dominated strategy is never optimal for a rational player.
- So it can be deleted before solving the game.
- Repeating this logic may dramatically simplify the game.

Warning

Many important games have no dominant strategy at all. Nash equilibrium is more general.

Prisoner's Dilemma: equilibrium need not be efficient

	C	D
C	$(2, 2)$	$(0, 3)$
D	$(3, 0)$	$(1, 1)$

Reading the matrix

- D strictly dominates C for both players.
- Therefore the unique Nash equilibrium is

$$(D, D).$$

But notice

$$(C, C) \succ (D, D)$$

for both players.
So Nash equilibrium is **not** the same thing as social efficiency or Pareto efficiency.

Iterated elimination of strictly dominated strategies

	L	R
U	$(4, 2)$	$(1, 1)$
M	$(3, 0)$	$(0, 2)$
D	$(2, 3)$	$(2, 0)$

Step-by-step logic

1. For Player 1, M is strictly dominated by U:
$$4 > 3, \qquad 1 > 0.$$
2. After deleting M, for Player 2, R is strictly dominated by L:
$$2 > 1, \qquad 3 > 0.$$
3. With only column L left, Player 1 prefers U to D:
$$4 > 2.$$

Outcome

After deleting M and then R, only
$$(U, L)$$
remains.

Why iterated elimination is useful

A different question from best responses

Best-response arrows ask: where are the Nash equilibria? Iterated elimination asks: which strategies can rational players rule out even before solving the whole game?

Why economists like it

- It uses only local comparisons, so it scales better in larger games.
- It often shrinks the game dramatically before equilibrium calculation.
- It highlights common knowledge of rationality, not just equilibrium consistency.

For this matrix

Here, both methods happen to find the same answer. In larger games, iterated elimination is often the faster and more informative first step.

Technology adoption as a coordination game

	New	Old
New	$(4, 4)$	$(0, 3)$
Old	$(3, 0)$	$(2, 2)$

Two pure Nash equilibria

$$(\text{New}, \text{New}), \qquad (\text{Old}, \text{Old}).$$

Reading the example

Think of two firms deciding whether to adopt a new industry standard. If both switch, compatibility creates high returns. If only one switches, that firm pays the switching cost alone.

Selection problem

The efficient outcome is (New, New), but (Old, Old) is safer when each firm doubts the other will switch.

Cournot revisited: from optimization to best response

Recall the 2-firm linear Cournot model

$$p = a - q_1 - q_2, \qquad MC = c.$$

Each firm solves

$$\max_{q_i \geq 0} \; (a - q_1 - q_2 - c)q_i.$$

Best-response functions

$$\mathrm{BR}_1(q_2) = \frac{a - c - q_2}{2}, \qquad \mathrm{BR}_2(q_1) = \frac{a - c - q_1}{2}.$$

Game-theoretic interpretation

A quantity choice is optimal only relative to a conjecture about the rival's quantity. That is why Cournot is a normal-form game.

Cournot equilibrium as an intersection of reaction curves

Nash condition

At equilibrium, each firm's quantity lies on its best-response function:

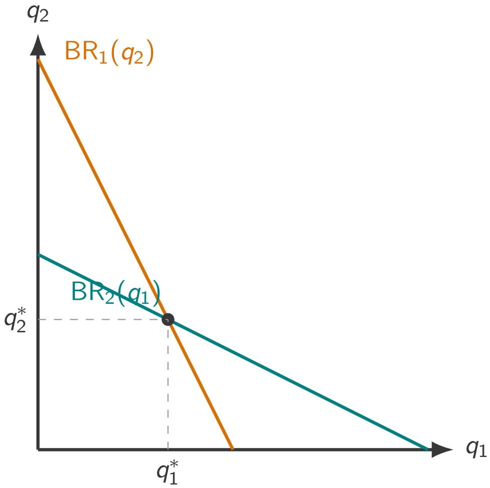
$$q_1^* \in \mathrm{BR}_1(q_2^*), \qquad q_2^* \in \mathrm{BR}_2(q_1^*).$$

Symmetric linear benchmark

For the 2-firm linear model,

$$q_1^* = q_2^* = \frac{a-c}{3}.$$

Each firm is choosing its profit-maximizing output given the rival's output.



Why pure strategies may fail: Matching Pennies

	H	T
H	$(1, -1)$	$(-1, 1)$
T	$(-1, 1)$	$(1, -1)$

teal: Player 1's best response   orange: Player 2's best response

How the game works

Player 1 prefers the two actions to match. Player 2 prefers the two actions to differ.

No pure Nash equilibrium

Whatever action profile we pick, one player wants to switch:

- if actions match, Player 2 wants to change;
- if actions differ, Player 1 wants to change.

Implication

To get a stable prediction, we must allow players to randomize. A probability distribution over actions is a **mixed strategy**.

Mixed strategies: definition

Definition

A mixed strategy assigns a probability to each pure action. If player i has two actions U and D, then

$$\sigma_i = (p, 1 - p)$$

means: choose U with probability p and choose D with probability $1 - p$, where $0 \leq p \leq 1$.

How to read the notation

For any pure action s_i, the number $\sigma_i(s_i)$ is simply the probability assigned to that action.

Key equilibrium rule

If a player puts positive probability on several actions in equilibrium, those actions must yield the **same expected payoff**. Otherwise the player would put zero weight on the worse one.

Expected payoffs under mixing

General formula

If players mix independently, the expected payoff is

$$U_i(\sigma_1, \ldots, \sigma_n) = \sum_{s \in S_1 \times \cdots \times S_n} \left(\prod_{j=1}^{n} \sigma_j(s_j) \right) u_i(s).$$

Here $\sigma_j(s_j)$ is the probability that player j assigns to pure action s_j.

2-by-2 example

If Player 1 uses p on U and Player 2 uses q on L, then

$$U_1(p, q) = pq\, u_1(U, L) + p(1-q)\, u_1(U, R) + (1-p)q\, u_1(D, L) + (1-p)(1-q)\, u_1(D, R).$$

Solving Matching Pennies by indifference

Suppose Player 2 chooses H with probability q and T with probability $1 - q$.

	$H\ (q)$	$T\ (1-q)$
$H\ (p)$	$(1, -1)$	$(-1, 1)$
$T\ (1-p)$	$(-1, 1)$	$(1, -1)$

Player 1's expected payoffs

$$U_1(H) = q - (1 - q) = 2q - 1,$$
$$U_1(T) = (1 - q) - q = 1 - 2q.$$

Player 1 is willing to mix only if

$$2q - 1 = 1 - 2q,$$

so

$$q = \frac{1}{2}.$$

By symmetry, if Player 1 chooses H with probability p, Player 2 is willing to mix only if

$$p = \frac{1}{2}.$$

Mixed Nash equilibrium

$$p^* = q^* = \frac{1}{2}.$$

Each player randomizes to keep the other indifferent.

Interpretation

The point of mixing is **unpredictability**. If one side became predictable, the other side would exploit it immediately.

Why mixed strategies matter beyond pure equilibrium

When should we care about mixed equilibrium?

- If no pure Nash equilibrium exists, mixed strategies rescue existence.
- In inspection, security, pricing, or sports settings, **unpredictability** is itself part of optimal behavior.
- Even if pure equilibria exist, additional mixed equilibria may coexist with them.

Nash's existence theorem

John Nash proved in 1950–1951 that every **finite** game has at least one Nash equilibrium once mixed strategies are allowed.
The proof uses a fixed-point theorem: in the space of probability choices, there must be a profile where each player's mixture is a best response to the others.

Historical role

This theorem turned game theory into a general theory of strategic interaction, not just a collection of special examples. Later refinements such as SPE keep Nash equilibrium as the benchmark and then impose extra discipline.

Mixed equilibrium: the general computation rule

Support-indifference principle

To solve a mixed equilibrium:

1. guess which pure strategies are used with positive probability,
2. choose probabilities that make the other player indifferent across that support,
3. check that strategies outside the support are not better.

Two important remarks

- Mixed equilibrium probabilities need *not* be $1/2$.
- A game may have both pure and mixed Nash equilibria.

When to start looking for one

Look for a mixed equilibrium when pure actions do not give a stable prediction, or when players have a reason to remain deliberately unpredictable.

A minimal support-check example

	L	R
U	$(2,2)$	$(0,0)$
M	$(0,1)$	$(0,1)$
D	$(0,0)$	$(2,2)$

Run all three steps

1. Guess the support: let Player 1 mix over $\{U, D\}$ and leave M out.
2. If Player 2 uses q on L, then

$$U_1(U) = 2q, \qquad U_1(D) = 2(1-q).$$

   Indifference gives $q = \frac{1}{2}$; by symmetry, $p = \frac{1}{2}$.

3. Check the excluded strategy:

$$U_1(M) = 0 < 1 = U_1(U) = U_1(D).$$

   So M should indeed get zero probability.

Mixed Nash equilibrium

$$\sigma_1 = \left(\frac{1}{2}, 0, \frac{1}{2}\right), \qquad \sigma_2 = \left(\frac{1}{2}, \frac{1}{2}\right).$$

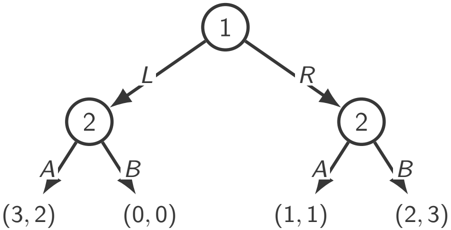
Extensive form: when timing is explicit

A game tree records

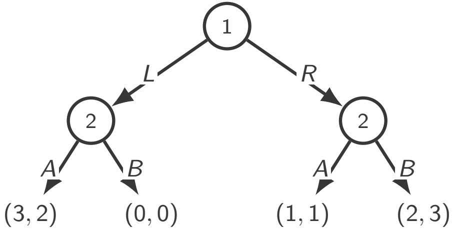
- who moves at each node,
- what actions are available,
- what players know when they move,
- and the payoffs at terminal nodes.

Perfect vs. complete information

- **Perfect information**: every player observes the full history of past actions.
- **Complete information**: players know the payoff structure and feasible actions.



A strategy in a tree is a complete contingent plan



Strategy sets

Player 1:

$$S_1 = \{L, R\}.$$

Player 2 must specify what to do after *both* histories:

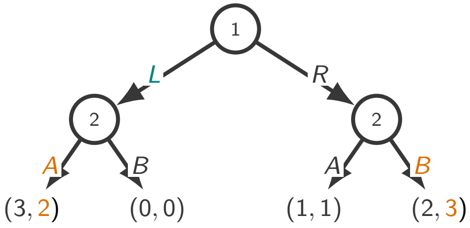
$$S_2 = \{AA, AB, BA, BB\}.$$

- AB means: choose A after L, choose B after R.
- Only one instruction is realized on path, but off-path instructions still matter for equilibrium analysis.

Why the tree is more informative

In normal form, Player 2's strategies AA, AB, BA, BB become four labels in a list. The matrix no longer visibly tells you which action is taken after L and which after R. The tree keeps those contingencies explicit, so timing and credibility are easier to see.

Backward induction on the tree



Solve from the end

After L, Player 2 chooses A because $2 > 0$. After R, Player 2 chooses B because $3 > 1$.

Then move one step back

Player 1 anticipates these continuation choices and compares

$$L \Rightarrow 3, \qquad R \Rightarrow 2,$$

so Player 1 chooses L.

Solution

$$\big(L;\;\; A \text{ after } L,\;\; B \text{ after } R\big)$$

with payoff $(3, 2)$. This is the same logic as Stackelberg.

Subgames, information sets, and SPE

Information set

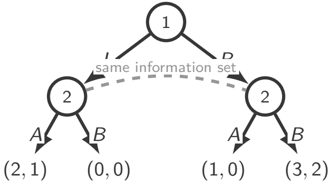
An information set is a collection of decision nodes that the mover cannot distinguish.

Subgame

A subgame starts at a decision node and includes all successor nodes. So it cannot start at just one node inside an information set.

SPE

SPE requires a Nash equilibrium in every subgame, so continuation actions must remain optimal after every history.



Why the dashed link matters

Here Player 2 does not know whether L or R occurred, so the left node alone is not a valid subgame. That is why SPE is stronger than Nash in dynamic settings: it rules out non-credible threats by checking every genuine continuation game.

Entry deterrence in normal form

As a normal-form game

	Accommodate	Fight
Out	$(0, 2)$	$(0, 2)$
In	$(1, 1)$	$(-1, -1)$

teal: Entrant's best response orange: Incumbent's best response

Mutual best responses occur at

$$(\text{Out}, \text{Fight}), \qquad (\text{In}, \text{Accommodate}).$$

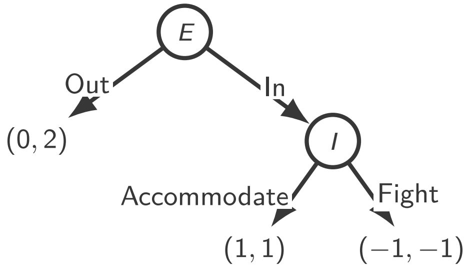
Why there are two Nash equilibria

- If the incumbent plans **Accommodate**, the entrant prefers **In**.
- If the incumbent plans **Fight**, the entrant prefers **Out**.
- If the entrant chooses **Out**, the incumbent is indifferent because both columns give payoff 2.

Normal-form prediction

$$(\text{Out}, \text{Fight}), \qquad (\text{In}, \text{Accommodate}).$$

Entry deterrence: credibility and SPE



Subgame logic

After entry, the incumbent compares

$$1 \quad \text{vs.} \quad -1,$$

so it strictly prefers **Accommodate**.

Why SPE refines Nash

A threat is **non-credible** if it helps support a Nash equilibrium ex ante but would not be optimal once the node is actually reached. Here "Fight" is non-credible. SPE keeps only those Nash equilibria whose continuation plans are optimal in every subgame.

Unique SPE

$$(\text{In}, \text{Accommodate}).$$

Stackelberg revisited as a subgame-perfect equilibrium

Follower's problem

Given leader output q_L, the follower chooses

$$q_F^*(q_L) \in \mathrm{BR}_F(q_L).$$

Leader's problem

Anticipating this reaction, the leader chooses

$$\max_{q_L \geq 0} \ \pi_L\big(q_L, q_F^*(q_L)\big).$$

What changed relative to Cournot?

- Same firms.
- Same demand and cost primitives.
- Different **timing**.

Key message

Changing the order of moves changes the game, hence the solution concept and often the equilibrium outcome.

Conceptual comparison

$$\text{Cournot} \Rightarrow \text{Nash}, \qquad \text{Stackelberg} \Rightarrow \text{SPE}.$$

Ultimatum bargaining: SPE logic

Baseline game

Player 1 proposes how to divide a pie of size 1:

$$(1 - x, x).$$

Player 2 then chooses **Accept** or **Reject**. If Player 2 rejects, both receive

$$(0, 0).$$

Outside option

The payoff from rejecting the current offer is called a player's **outside option** or **disagreement payoff**. In this baseline ultimatum game, that outside option is 0 for both players.

Backward-induction logic

Since any positive x is better than 0, Player 2 accepts any positive offer. Anticipating this, Player 1 offers the smallest amount that will be accepted.

Nash vs. SPE

In a normal-form description, strange Nash equilibria can be supported by threats like "I will reject any offer below 0.9." But once such an offer is actually on the table, rejecting a positive amount is not optimal. SPE removes these non-credible rejection threats and keeps the sequentially rational outcome.

Outside options, continuation values, and future play

If rejection gives (d_1, d_2)

Player 2 accepts whenever

$$x \geq d_2.$$

So Player 1 cannot push the offer below Player 2's outside option.

Continuation value

A **continuation value** is the payoff a player expects if agreement is delayed and the game moves on. That is just another kind of outside option: it is what you get from *not* accepting now.

Economic meaning

Better outside options make a player harder to exploit in bargaining. They improve what the player can credibly insist on.

Bridge to repeated games

In repeated interaction, current actions depend on future continuation values. That is why future punishment and reward can discipline behavior today.

Repeated games: why one-shot logic is incomplete

In a repeated game

The same stage game is played again and again.

- A strategy now maps **histories** into current actions.
- Today's action affects tomorrow's continuation payoff.
- Punishment and reward become possible.

Main implication

Outcomes that are impossible in the one-shot game can sometimes be sustained when players care enough about the future.

Intuition

"If you deviate today, I respond tomorrow." That is exactly the kind of logic absent in a one-shot normal-form game.

Repeated Prisoner's Dilemma and grim trigger

Stage-game payoffs

	C	D
C	(R, R)	(S, T)
D	(T, S)	(P, P)

with $T > R > P > S$, where T is temptation, R is mutual cooperation, P is mutual defection, and S is the sucker payoff.

Grim trigger

Cooperate unless someone defects. If a defection occurs, play D forever after.

Compare continuation values

If a player cooperates forever:

$$V_C = \frac{R}{1-\delta}.$$

If the player deviates once and is punished forever:

$$V_D = T + \frac{\delta P}{1-\delta}.$$

Sustainability condition

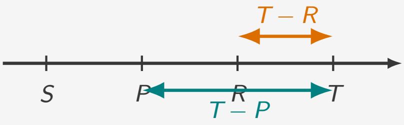
Cooperation can be sustained if

$$V_C \geq V_D \quad \Longleftrightarrow \quad \delta \geq \frac{T-R}{T-P}.$$

δ measures patience: future cooperation must outweigh the one-shot temptation to defect.

Visual intuition: temptation vs. punishment

Read the payoff line from left to right



Economic meaning

Starting from mutual cooperation at R, deviation gives the one-shot jump to T.
If grim trigger starts, the continuation path drops to P forever. So cooperation is harder to sustain when the temptation gap $T - R$ gets larger or the punishment gap $T - P$ gets smaller.

Why grim trigger is subgame-perfect

On the cooperative path

No one deviates exactly when

$$V_C \geq V_D.$$

After any deviation

The continuation is D forever. Since D is a best response to D in the stage game (because $P > S$), once punishment starts no player wants to switch back unilaterally to C.

SPE conclusion

Grim trigger is an SPE when

$$\delta \geq \frac{T - R}{T - P}.$$

Future cooperation must be valuable enough to outweigh the one-shot temptation to defect.

Oligopoly application: collusion is a repeated-game issue

Mechanism

In one-shot Bertrand or Cournot competition, each firm wants to undercut or expand output.
With repeated interaction, future punishment can deter today's deviation.

What supports collusion?

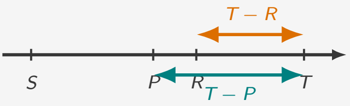
- large δ: firms care more about future cartel profits,
- small $T - R$ and large $T - P$: deviation gains are limited relative to punishment losses,
- clear monitoring: low prices are quickly interpreted as cheating,
- stable demand and costs: unusual prices are easier to read as cheating,
- simple punishment plans: everyone knows when and how to retaliate.

Why many firms make collusion fragile: payoff incentives

Two incentive effects when there are many firms

1. Under collusion, each firm receives only its cartel share. With many firms, that share is small, but a secret price cut or output expansion can attract customers from many rivals at once. The current-period gain from cheating can therefore be large.
2. If punishment means a price war or a return to noncooperative play, the deviator loses future cartel profits. But with many firms those future profits were already split across many participants, so the loss from triggering punishment is smaller.

Payoff-line intuition



Many firms can push T far right while leaving R and P relatively close, so $T - R$ is large but $T - P$ is only slightly larger. The critical δ is then high.

Why many firms make collusion fragile: information and coordination

3. Trigger strategies require signal extraction

Collusion often relies on a trigger strategy: if cheating is detected, firms switch to punishment. That requires them to infer whether someone deviated, who deviated, and whether punishment should start now.

With many firms, a lower market price, a fall in your own sales, or a change in total quantity is ambiguous. The same signal could reflect one firm's secret discount, a demand shock, a cost shock, or several firms adjusting at once.

4. Punishment must also be coordinated

Deterrence requires more than "everyone wants to punish." Firms must actually carry out the punishment.

Who cuts price first? Will everyone follow? Will some firms free-ride and let others punish? The more firms there are, the harder it is to coordinate these actions, so the punishment threat becomes less credible.

End of Lecture 4

- A game is about strategic interdependence: each player's payoff depends on others' choices.
- In simultaneous-move games, Nash equilibrium means mutual best responses.
- In sequential games, backward induction and SPE impose sequential rationality and rule out non-credible threats.
- Bargaining and repeated games show how timing, commitment, and future punishment reshape outcomes.

Core distinctions to remember

$$\text{dominance} \neq \text{best response} \neq \text{efficiency},$$

$$\text{Nash} \neq \text{SPE whenever credibility matters.}$$

Next lecture

Where we are going

So far everyone knows the game being played. Next we allow players to have different information.

- Bayesian games and Bayesian Nash equilibrium
- adverse selection vs. moral hazard
- signaling and screening as responses to information asymmetry

Bridge

Chapter 3 asked: "What if my payoff depends on your action?"
Chapter 4 asks: "What if my action also depends on what I know and what I think you know?"

References

Selected references for Lecture 4

- Mas-Colell, A., Whinston, M. D., and Green, J. R. (1995). *Microeconomic Theory*. Oxford University Press, esp. Chs. 7–9.
- Osborne, M. J. (2004). *An Introduction to Game Theory*. Oxford University Press.
- Nash, J. (1951). "Non-cooperative Games." *Annals of Mathematics*.
- Fudenberg, D., and Tirole, J. (1991). *Game Theory*. MIT Press.
- Jehle, G. A., and Reny, P. J. (2011). *Advanced Microeconomic Theory*. 3rd ed. Financial Times/Prentice Hall, esp. Chs. 8–9.