Advanced Microeconomics

Lecture 2: Lagrangian Method, Duality & Markets (Part I)

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Outline

1. Finish Chapter 1: revealed preference, duality, Slutsky, welfare
2. Lagrangian method (multi-good) and the meaning of multipliers
3. Start Chapter 2: markets and price mechanisms (partial equilibrium)

Where we are in Chapter 1

Last lecture

Preferences $\Rightarrow$ rational choice $\Rightarrow$ utility representation.

Today

- How to *infer* rationality from observed choices (revealed preference).
- How to *solve* consumer problems systematically (Lagrangian method).
- Duality: utility maximization vs. expenditure minimization.
- Slutsky decomposition and welfare measures (CV/EV).

Revealed preference: why we need it

Utility is a representation device. In data we often observe:

$$(p^t, m^t, x^t) \quad \text{(prices, income, chosen bundle).}$$

Question

When can observed choices come from *some* rational preference?

Message

Revealed preference axioms provide testable restrictions on demand data.

WARP (Weak Axiom of Revealed Preference)

Directly revealed (weak) preference

If x^t is chosen at (p^t, m^t) and y is affordable at the same budget,

$$p^t \cdot y \leq m^t,$$

then we say x^t is *directly revealed weakly preferred* to y and write $x^t R y$.

Directly revealed *strict* preference

If x^t is chosen at (p^t, m^t) and y is *strictly* affordable,

$$p^t \cdot y < m^t,$$

then we say x^t is *directly revealed strictly preferred* to y and write $x^t P y$.

WARP (a clean statement using strict revealed preference)

It cannot happen that $x^t P x^s$ and $x^s P x^t$ for two observations $t \neq s$. (Strict two-cycle is ruled out.)

A simple WARP check (two observations)

Observation t: prices p^t, chosen bundle x^t.
Observation s: prices p^s, chosen bundle x^s.

Compute two affordability tests

$$\text{Is } x^s \text{ strictly affordable at } (p^t, m^t)? \quad p^t \cdot x^s < m^t,$$

$$\text{Is } x^t \text{ strictly affordable at } (p^s, m^s)? \quad p^s \cdot x^t < m^s.$$

WARP violation

If both are affordable, then each choice "reveals" preference against the other: a contradiction.

What "weak" means

WARP rules out **two-point (pairwise) cycles**. Stronger axioms (SARP/GARP) rule out **longer cycles** and are used for full rationalizability results.

Utility maximization in n goods

Let $x \in \mathbb{R}^n_+$, prices $p \in \mathbb{R}^n_+$, income $m > 0$.

Utility maximization problem (UMP)

$$\max_{x \in \mathbb{R}^n_+} \; u(x) \quad \text{s.t.} \quad p \cdot x \leq m.$$

Why we need a method

In n dimensions, "substitution" is not a simple one-variable plug-in. We need a systematic way to handle constraints.

The Lagrangian: setup

Lagrangian

$$\mathcal{L}(x,\lambda) = u(x) + \lambda\,(m - p\cdot x), \qquad \lambda \geq 0.$$

First-order conditions (interior; $x_i > 0$)

$$\frac{\partial u(x)}{\partial x_i} = \lambda p_i \quad (i = 1,\dots,n), \qquad p\cdot x = m.$$

Economic content

At the optimum, marginal utility per dollar is equalized across goods:

$$\frac{\partial u/\partial x_i}{p_i} = \lambda \quad \forall i.$$

What does the multiplier λ mean?

Fix prices p and consider how the *optimal value* changes with income m.

Value function and Envelope intuition

$$V(m) \equiv \max_{x \in \mathbb{R}^n_+} \{u(x) : p \cdot x \leq m\}.$$

At the optimum, a small increase in m relaxes the constraint by one unit, so

$$\frac{dV(m)}{dm} = \lambda^*(p, m).$$

Interpretation

λ^* is the **marginal utility of income**: how much the optimal utility increases when income increases by one unit.

Rule of thumb

Large λ^* means the consumer is "tight" (income is valuable); small λ^* means the budget is slack (rare under monotonicity).

Corner solutions and Kuhn–Tucker (only the idea)

In general, some goods may not be consumed: $x_i^* = 0$.

KKT logic (informal)

$$\frac{\partial u(x^*)}{\partial x_i} \le \lambda^* p_i, \quad \text{with equality if } x_i^* > 0.$$

Interpretation

If a good is not purchased, its "marginal utility per dollar" is not high enough relative to the common level λ^*.

A worked example in n goods: Cobb–Douglas

Let $u(x) = \prod_{i=1}^{n} x_i^{\alpha_i}$ with $\alpha_i > 0$ and $\sum_i \alpha_i = 1$.

Result (Marshallian demand)

$$x_i^*(p, m) = \alpha_i \frac{m}{p_i} \qquad (i = 1, \dots, n).$$

Multiplier

Using $\frac{\partial u}{\partial x_i} = \lambda p_i$ implies

$$\lambda^*(p, m) = \frac{u(x^*)}{m}.$$

Since u is homogeneous of degree 1 (here $\sum_i \alpha_i = 1$), scaling income scales the optimal bundle: $x^*(p, 2m) = 2x^*(p, m)$, hence $u(x^*(p, 2m)) = 2u(x^*(p, m))$. Therefore $\lambda^*(p, m) = u(x^*)/m$ is *independent of* m under this normalization.

A worked example in n goods: Cobb–Douglas

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Multiplier

For $u(x) = \prod_{i=1}^{n} x_i^{\alpha_i}$ with $\sum_i \alpha_i = 1$,

$$\frac{\partial u}{\partial x_i} = \alpha_i \frac{u(x)}{x_i}.$$

FOC gives $\alpha_i \frac{u(x^*)}{x_i^*} = \lambda^* p_i$, i.e. $p_i x_i^* = \frac{\alpha_i}{\lambda^*} u(x^*)$. Summing over i and using $\sum_i \alpha_i = 1$,

$$m = \sum_i p_i x_i^* = \frac{u(x^*)}{\lambda^*} \;\Rightarrow\; \lambda^*(p, m) = \frac{u(x^*)}{m}.$$

Duality: two problems, one structure

UMP (choose best bundle given money)

$$v(p, m) = \max_{x \in \mathbb{R}^n_+} \{u(x) : p \cdot x \leq m\}, \quad x(p, m) \text{ (Marshallian demand).}$$

EMP (choose cheapest bundle given target utility)

$$e(p, \bar{u}) = \min_{x \in \mathbb{R}^n_+} \{p \cdot x : u(x) \geq \bar{u}\}, \quad h(p, \bar{u}) \text{ (Hicksian demand).}$$

Key dual relationships

Two identities

$$v\big(p,\ e(p,\bar{u})\big) = \bar{u}, \qquad e\big(p,\ v(p,m)\big) = m.$$

Interpretation

UMP tells you the *best utility you can reach* with income m. EMP tells you the *minimum income you need* to reach utility $\bar{u}$. They are inverse mappings.

Lagrange multiplier in EMP (shadow price of utility)

Consider EMP with constraint $u(x) \geq \bar{u}$.

Lagrangian

$$\mathcal{L}(x, \mu) = p \cdot x + \mu(\bar{u} - u(x)), \qquad \mu \geq 0.$$

FOC (interior)

$$p_i = \mu \frac{\partial u(x)}{\partial x_i} \quad \forall i.$$

Interpretation

μ^* is the **marginal expenditure needed to raise utility**:

$$\frac{\partial e(p, \bar{u})}{\partial \bar{u}} = \mu^*(p, \bar{u}).$$

Slutsky Equation: Introduction (1/4)

Recall: two demand concepts

- Marshallian demand $x(p, m)$ solves
$$\max_x \{u(x) : p \cdot x \leq m\}.$$
- Hicksian (compensated) demand $h(p, \bar{u})$ solves
$$\min_x \{p \cdot x : u(x) \geq \bar{u}\}.$$

Key question

When a price changes (say p_j), how do we decompose the total effect on x_i into a substitution effect and an income effect?

Slutsky Equation: Step-by-Step (2/4)

Thought experiment

Let the price of good j change from p_j to $p_j + dp_j$. We want to understand how x_i changes.

Step 1: isolate substitution (compensate to keep utility fixed)

Let $u^* = v(p, m)$ be the initial achieved utility. Imagine we compensate the consumer so they can still reach u^* after the price change. Then look at how $h_i(p, u^*)$ responds to p_j:

$$\text{substitution effect} = \frac{\partial h_i(p, u^*)}{\partial p_j}.$$

Slutsky Equation: Step-by-Step (3/4)

Step 2: add the income (purchasing power) effect

Without compensation, the price increase reduces real purchasing power. This additional effect is captured by

$$-x_j(p,m)\,\frac{\partial x_i(p,m)}{\partial m}.$$

Slutsky equation

$$\frac{\partial x_i(p,m)}{\partial p_j} = \underbrace{\frac{\partial h_i(p,u^*)}{\partial p_j}}_{\text{substitution (Hicksian) effect}} \underbrace{-\,x_j(p,m)\,\frac{\partial x_i(p,m)}{\partial m}}_{\text{income effect}}.$$

Interpretation

First term: behavior change when utility is held constant.
Second term: additional change because real income shifts.

Slutsky Equation: Why It Matters (4/4)

Policy relevance

If a tax changes a price, Slutsky tells us how much of the demand response comes from *pure substitution* vs. *purchasing power*.

Sign intuition

- Substitution effect for the price-increasing good is typically negative.
- Income effect can be negative (normal goods) or positive (inferior goods).

Why Slutsky matters in practice: from observed demand to compensated demand

What we observe vs. what we need

In data, we typically observe how choices change with **prices and income**: this identifies (or lets us estimate) **Marshallian demand** $x(p, m)$. We do *not* directly observe choices under a hypothetical experiment that keeps utility constant.

What Slutsky provides

Slutsky links the **observed total price effect** to the **compensated (Hicksian) price effect**. So, once we know/estimate the Marshallian price and income derivatives, we can recover the **pure substitution response** (the compensated slope).

Welfare and cost-of-living analysis

The expenditure function $e(p, \bar{u})$ and Hicksian demand $h(p, \bar{u})$ are the key ingredients for **cost-of-living indices** and welfare measures such as **CV** and **EV**.

Introducing CV and EV

Motivation

When prices change (e.g., from p^0 to p^1), we want to measure:

- How much better or worse off is the consumer?
- What is the money value of this price change?

Key difference

Both CV and EV measure welfare changes, but use different reference points (new prices vs. old prices).

Message

CV and EV refine welfare analysis by translating utility changes into money units.

Welfare Measure: Compensating Variation (CV)

Let $u^0 = v(p^0, m)$ be the initial utility.

Definition

$$CV = e(p^1, u^0) - e(p^0, u^0).$$

Interpretation

CV is the additional income needed *at the new prices* p^1 to reach the original utility level u^0.

How to read it

"How much compensation is needed after the price change to keep the consumer as well off as before?"

Welfare Measure: Equivalent Variation (EV)

Let $u^1 = v(p^1, m)$ be the utility after the price change.

Definition

$$EV = e(p^0, u^0) - e(p^0, u^1).$$

Interpretation

EV is the amount of income that, *at the original prices* p^0, would reduce utility from u^0 down to the new level u^1.

How to read it

"How much would the consumer be willing to pay beforehand to avoid the price change?"

Summary: demand and welfare objects

- Marshallian demand $x(p, m)$ vs. Hicksian demand $h(p, \bar{u})$.
- Slutsky equation: total effect = substitution effect + income effect.
- Expenditure function $e(p, \bar{u})$ as the bridge from utility to money.
- Welfare measures: CV and EV for monetizing price changes.

Takeaway

Duality $\rightarrow$ Slutsky $\rightarrow$ CV/EV: the same structure reappears in many policy applications.

Why study markets and prices?

- Most resource allocation in real economies is mediated by markets and prices.
- Policy instruments (taxes, subsidies, regulations) work largely through price and quantity responses.

Goal of Chapter 2 (Part I today)

Build the partial-equilibrium toolkit: supply, demand, equilibrium, and basic policy comparative statics.

Supply, demand, and partial equilibrium

Demand and supply functions

$$Q^D = D(p), \qquad Q^S = S(p).$$

Partial equilibrium

Study one market "in isolation": take other prices and incomes as given.

Key question

What determines the equilibrium price p^* and how does it change under shocks and policies?

Equilibrium condition

$$D(p^*) = S(p^*).$$

Two objects

- p^*: equilibrium price.
- $Q^* = D(p^*) = S(p^*)$: equilibrium quantity.

Rule of thumb

Equilibrium is a *fixed point*: the price that makes planned purchases equal planned sales.

Shifts vs. movements

Movement along a curve

Price changes $\Rightarrow$ quantity demanded/supplied changes along the same curve.

Shift of a curve

Non-price determinants change $\Rightarrow$ the entire demand or supply curve shifts.

Examples

Demand shifters: income, tastes, prices of substitutes/complements.
Supply shifters: input costs, technology, taxes on production.

Policy comparative statics: a per-unit tax on sellers

Suppose a per-unit tax τ is imposed on producers.

Effective supply

If producers receive net price $p - \tau$, then equilibrium satisfies

$$D(p) = S(p - \tau).$$

Tax incidence

The division of the burden between buyers and sellers depends on elasticities: more inelastic side bears more of the burden.

Price controls: ceilings and floors

Price ceiling $\bar{p} < p^*$

Creates excess demand (shortage): $D(\bar{p}) > S(\bar{p})$.

Price floor $\underline{p} > p^*$

Creates excess supply (surplus): $S(\underline{p}) > D(\underline{p})$.

Interpretation

Price controls change *which side is rationed* and *how much trade is lost* relative to the competitive benchmark.

End of Lecture 2

- Revealed preference: WARP as a consistency test for demand data.
- Lagrangian method in n goods; meaning of multipliers (λ and μ).
- Duality: UMP vs. EMP; $v(p, m)$ and $e(p, \bar{u})$.
- Slutsky decomposition; welfare measures CV and EV.
- Market basics: partial equilibrium, shifts vs. movements, taxes, and price controls.

Next lecture

Finish Chapter 2: perfect competition, monopoly, and basic oligopoly comparisons.

References

References for Lecture 2 (revealed preference, duality, Slutsky; partial equilibrium)

- Varian, H. R. (1992). *Microeconomic Analysis*. Chapters on revealed preference and duality.
- Mas-Colell, A., Whinston, M. D., and Green, J. R. (1995). *Microeconomic Theory*. Consumer theory chapters (revealed preference, duality, Slutsky).
- Jehle, G. A., and Reny, P. J. (2011). *Advanced Microeconomic Theory*. Consumer demand and welfare measures.
- For partial equilibrium tools: any standard intermediate text; we will use them as a stepping stone to market structures in Chapter 2.